Lecture 6:

Strong Marcov Property, Decomposition Theorem

Last Time

Let $\{X_t\}_{t\in\mathbb{N}}$ be a time homogeneous Markov chain, $\mathcal{X}$ be its state space, P be the transition matrix such that $P_{xy} = \mathbb{P}(X_1 = y \mid X_0 = x)$ is the transition probability from the state x to y.

0.1 Define: $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot \mid X_0 = x)$,

$\mathbb{E}_x(\cdot) = \mathbb{E}(\cdot \mid X_0 = x)$.

the time first visit to x:

$$\tau_x = \min\{n \geq 1 \mid X_n = x\} =: \tau_x^1$$

the time of kth visit to x:

$$\tau_x^k = \min\{n > \tau_x^{k-1} \mid X_n = x\}, \quad \forall k \geq 2.$$

$$\rho_{xy} = \mathbb{P}(\tau_y < \infty \mid X_0 = x) = \mathbb{P}_x(\tau_y < \infty)$$

e.g., $\rho_{yy} = \mathbb{P}_y(\tau_y < \infty)$

0.2 A state x communicates with a state y if $[P^n]_{xy} > 0$ for some $n \geq 1$, which is denoted by $x \longrightarrow y$.

0.3 Lemma 1. $x \longrightarrow y$ iff $\rho_{xy} > 0$.

0.4 Lemma 2. (Transitivity) $x \to y$ & $y \to z \Rightarrow x \to z$.

0.5 A state x is called transient if $\rho_{xx} < 1$.

A state x is called recurrent if $\rho_{xx} = 1$.

0.6 Thm 1. $x \to y$ & $\rho_{yx} < 1 \Rightarrow x$ is transient.

0.7 Cor 1. $x \to y$ & x is recurrent $\Rightarrow \rho_{yx} = 1$.

$\boxed{\text{TODAY}}$ 1°. Lemma 3. If $\exists\, k > 1$, $\alpha > 0$, such that $\forall\, x \in \mathcal{X}$

$\mathbb{P}_x(\tau_y \le k) \ge \alpha$, then $\mathbb{P}_x(\tau_y = \infty) = 0$, $\forall\, x \in \mathcal{X}$.

(i.e. $\rho_{xy} = 1 \ \forall\, x \in \mathcal{X}$). In particular, y is recurrent.

Pf. Since $\mathbb{P}_x(\tau_y > k) \le 1 - \alpha$, by mathematical induction,

why?

$$\mathbb{P}_x(\tau_y > nk) \le (1-\alpha)^n.$$

Taking limits at both sides yields

why?

$$\mathbb{P}_x(\tau_y = \infty) = \lim_{n\to\infty} \mathbb{P}_x(\tau_y > nk) \le \lim_{n\to\infty} (1-\alpha)^n = 0.$$

Thus, $\mathbb{P}_x(\tau_y = \infty) = 0$, $\forall\, x \in \mathcal{X}$.

This implies $\rho_{xy} = 1 - \mathbb{P}_x(\tau_y = \infty) = 1$. $\blacksquare$

2°. Def We say that T is a stopping time if the occurrence (or nonoccurrence) of the event "we stop at time n", $\{T=n\}$, can be determined by looking at the values of the process up to that time: $X_0, X_1, \cdots, X_n$.

Ex1. For a Markov chain $(X_n)_{n\in\mathbb{N}}$, to see that T_y is a stopping time, note that

$$\boxed{\{T_y = n\} = \{X_1 \neq y, X_2 \neq y, \cdots, X_{n-1} \neq y, X_n = y\}}$$

and that the right hand side can be determined from $X_0, X_1, X_2, \cdots, X_n$.

Ex2. For a random walk on $\mathbb{Z}$ with 0.6/0.4 probability to move a step right/left. Let S be the final time that integer 0 is ever visited by the chain. Note that $\{S=n\}$ encodes information about the future and thus it is not a stopping time.

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Theorem 2. (Strong Markov Property).

Suppose T is a stopping time. Given that $T=n$ and $X_T = y$, any other information about $X_0, \cdots, X_T$ is irrelevant for predicting the future. And $\{X_{T+k}\}_{k\in\mathbb{N}}$ behaves like the Markov chain with initial state y.

Proof. It is sufficient to show, $\forall n \in \mathbb{N},\ y, z \in \mathcal{X}$,

$$\boxed{\mathbb{P}(X_{T+1} = z \mid X_T = y,\ T=n) = P_{yz}.}$$

Here $\{X_T = y,\ T=n\}$ repesents a set of event.

Define $V_n := \left\{ (x_0, x_1, \cdots, x_n) \,\middle|\, \begin{array}{l} \text{If } X_0 = x_0, X_1 = x_1, \cdots, X_n = x_n, \\ \text{then } T=n \text{ and } X_T = y \end{array} \right\}.$

Thus, $\mathbb{P}(X_{T+1} = z,\ X_T = y,\ T=n)$

$$= \sum_{x=(x_0,x_1,\cdots,x_n)\in V_n} \mathbb{P}(X_{n+1} = z, X_n = x_n, X_{n-1} = x_{n-1}, \cdots, X_0 = x_0)$$

Multiplication Rule

$$= \sum_{x\in V_n} \mathbb{P}(X_{n+1} = z \mid X_n = x_n, \cdots, X_0 = x_0) \cdot \mathbb{P}(X_n = x_n, \cdots, X_0 = x_0)$$

Markov property

$$\overset{\text{Markov property}}{=} \sum_{x \in V_n} \mathbb{P}(X_{n+1} = z \mid X_n = x_n) \cdot \mathbb{P}(X_n = x_n, \cdots, X_0 = x_0)$$

$X_n = y$

$$\overset{X_n = y}{=} \sum_{x \in V_n} P_{yz} \cdot \mathbb{P}(X_n = x_n, \cdots, X_0 = x_0)$$

$$= P_{yz} \cdot \sum_{x \in V_n} \mathbb{P}(X_n = x_n, \cdots, X_0 = x_0)$$

$$= P_{yz} \cdot \mathbb{P}(T = n, X_T = y).$$

Therefore,

$$P_{yz} = \frac{\mathbb{P}(X_{T+1} = z, X_T = y, T = n)}{\mathbb{P}(T = n, X_T = y)}$$

$$= \mathbb{P}(X_{T+1} = z \mid X_T = y, T = n). \quad \blacksquare.$$

Remark 1. Recall $\tau_x = \min\{n \geq 1 \mid X_n = x\} := \tau_x^1$,

$\tau_x^k = \min\{n > \tau_x^{k-1} \mid X_n = x\}$, $\forall k \geq 2$.

Then the Strong Markov Property implies

$$\boxed{\mathbb{P}_x(\tau_y^k < \infty) = \rho_{xy} \cdot \rho_{yy}^{k-1}, \quad \forall k \geq 1, \forall x, y \in \mathcal{X}.}$$

In particular, $\boxed{\mathbb{P}_y(\tau_y^k < \infty) = \rho_{yy}^k.}$

(i). $\rho_{yy} = 1$; the probability of returning k times $\rho_{yy}^k = 1$, so the chain returns to y infinite many times. In this case, y is called recurrent, which continually recurs in the Markov chain.

(ii). $\rho_{yy} < 1$: the probability of returning k times $\rho_{yy}^k \longrightarrow 0$ as $k \to \infty$. In this case, the state y is called transient, since after some point it is never visited by the chain.

3°. Ex3. (A Seven-State Chain)

Consider the following transition probability:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0.7 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.3 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0.6 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Q: Identify the states that are recurrent and those that are transient.

A: Draw a graph which contains an arc with arrow from the state x to y if $P_{xy} > 0$ and $x \neq y$.

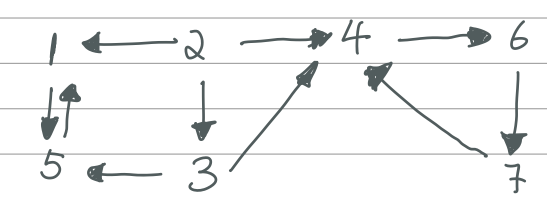

Figure 1

Lemma 4. Let $x, y \in \mathcal{X}$ be two distinct states. Then $x \longrightarrow y$ iff x reaches y from a sequence of arrows "$\longrightarrow$".

Ex3 (cont.) One can see that the state 2 communicates with 1 (i.e. $2 \longrightarrow 1$. Note here $\longrightarrow$ is not the same as $\longrightarrow$), since

$$P_{21} = \mathbb{P}_2(\tau_1 < \infty) \geq \mathbb{P}_2(\tau_1 = 1) = \mathbb{P}_2(X_1 = 1) = P_{21} > 0.$$

However, the state 1 does NOT communicate with state 2. ($P_{12} = 0 < 1$). So Theorem 1 implies that 2 is transient. Similarly, 3 is transient.

To see the rest of the states are recurrent, we need the following definition and a theorem.

4°. Def. A set A is closed if, for any $x \in A$ and $y \notin A$, $P_{xy} = 0$. (i.e., $x \not\to y$, $\forall x \in A$, $\forall y \notin A$).

A set B is irreducible if, for any $x, y \in B$, x communicates with y (i.e. $x \longrightarrow y$, $\forall x, y \in B$).

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A communicating class is a maximal irreducible set.

A set $C \subseteq \mathcal{X}$ is called a communicating class if

① $\forall\, x, y \in C$, $x \to y$ and $y \to x$.

② $\forall\, x \in C$, $y \notin C$, either $x \not\to y$ or $y \not\to x$.

Remark 2. For example, in Ex 3,

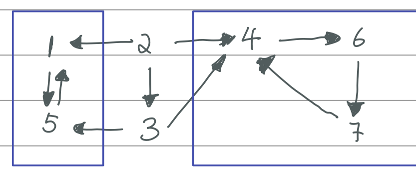

Figure 2

$\{1,5\}$ and $\{4,6,7\}$ are closed sets, their union $\{1,5,4,6,7\}$ is also closed. Moreover, adding $\{3\}$ provides another closed set $\{1,3,4,5,6,7\}$. Finally, the whole state space is always closed.

Among all these closed sets, only $\{1,5\}$ and $\{4,6,7\}$ are irreducible.

The communicating classes are $\{1,5\}$, $\{2\}$, $\{3\}$, and $\{4,6,7\}$.

Theorem 3. If C is a finite closed and irreducible set, then all states in C are recurrent.

Remark 3. Returning to Ex 1, Theorem 3 tells that states $1, 4, 5, 6, 7$ are recurrent, which completes the example.

Together with Theorem 1, Theorem 3 implies the following theorem.

Theorem 4. (Decomposition Theorem).

If the state space $\mathcal{X}$ is finite, then $\mathcal{X}$ can be written as a disjoint union $S \cup R_1 \cup \cdots \cup R_k$, where S is a set of transient sates and the R_i, $1 \le i \le k$, are closed irreducible sets of recurrent states.

This is the end of this lecture!

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